In search of local degrees of freedom in quadratic diff-invariant Lagrangians

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We show that local diff-invariant free field theories in four spacetime dimensions do not have local degrees of freedom.

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I. INTRODUCTION

There is a clear consensus nowadays about the most promising strategies to find a consistent theory of quantum gravity. An important majority of the leading theoretical physicists (and their followers) consider that superstring theories and their derivatives, especially M theory[1] hold the key to understanding general relativity in the quantum regime. This does not mean that there are no other potentially successful approaches to tackle this problem such as perturbative quantum gravity, quantum field theory (QFT) in curved backgrounds, Euclidean quantum gravity, Regge calculus, and lattice techniques, twistor theory, noncommutative geometry, nonperturbative quantum gravity, and many others [4]. Some of the most recent approaches, such as Ashtekar's nonperturbative formulation of gravity [5] and the general setting provided by the loop variables formalism to deal with diff-invariant theories [6] have received a lot of attention in recent years as very promising ways to tame the deep problems presented by the quantization of general relativity. There is also a clear consensus about the failure of perturbative formulations for gravity although there have been some recent and interesting papers on the subject [7]; it is thus curious to look at a series of papers [8,9] in the late 1980's and early 1990's[2] that give a novel understanding on this issue. In Ref. [8] Witten shows that $2+1$ gravity has a renormalizable perturbation expansion and gives the following explanation about the nonrenormalizability in the $(3+1)$-dimensional case: ''It is amusing to think about $(3+1)$-dimensional gravity from this point of view. The Lagrangian is of the general form

$$I_{(4)}\sim\int e\wedge e\wedge(d\omega+\omega\wedge\omega). \tag{3.13}$$

If one hopes for ''power-counting renormalizability,'' one needs to assign dimension one to both e and ω, so that every term in Eq. (3.13) is of dimension 4. (Again, this is in contrast to the fact that the metric and vierbein are usually considered to have dimension zero). As e and ω have positive dimension, the short-distance limit must have $e=\omega=0$. The problem is now that as Eq. (3.13) has no quadratic term in an expansion around $e=\omega=0$, one cannot make sense of the ''unbroken phase'' that should govern the short-distance behavior; that is the essence of the unrenormalizability of quantum gravity in four dimensions.''

Following this suggestion Deser, McCarthy, and Yang [9] analyzed the same problem by using a Palatini action. The advantage of their approach is that while preserving power counting renormalizability in $D=1,\ldots,4$ the action has a quadratic term in an expansion about zero field value. The main result of that paper, in the $(3+1)$-dimensional context, was to show that nonrenormalizability could be traced back to the mismatch between the symmetries of the full action and those of the kinetic term. Specifically, they showed that the quadratic part of the action had more independent gauge symmetries than the full action and hence one could not render the kinetic part invertible by using only the maximum number of gauge fixing conditions allowed by the symmetry of the full Lagrangian. The authors of that paper wonder ''Whether a viable local modification of gravity which exploits this ''near miss'' exists is an open question.''

These are several conceivable ways to advance in this direction such as looking for other actions for general relativity—with quadratic terms that can hopefully be inverted after gauge fixing—or modify the theory to provide it with kinetic terms with the desired properties.

A nice way to do this would be to add a diff-invariant quadratic kinetic term written in terms of the fields appearing in the action

$$S=\int e\wedge e\wedge F(\omega), \tag{1}$$

where e is a tetrad one form and $F(\omega)$ the curvature of a SO(3,1) spin connection[3] ω. If this is to succeed the quadratic part of the action should have, at least, two physical degrees of freedom because if it had less then the kinetic term would have more gauge symmetry than the full action and hence by gauge fixing the symmetries present in Eq. (1) one would not get an invertible kinetic term. Also some matching of the symmetries of S and the kinetic term should be imposed. With this philosophy in mind one is naturally led to pose the following question: Can a diff-invariant quadratic Lagrangian have local degrees of freedom in four dimensions? The purpose of this paper is to answer this ques-

[1]See Ref. [1] for a down-to-earth presentation on the subject or Refs. [2, 3] and references therein for a more complete treatment.

[2]One of us (J.F.B.G.) is grateful to Abhay Ashtekar for drawing his attention to these papers.

[3]Notice that this is something conceptually similar to the introduction of higher derivative terms but in the opposite direction.

tion. We want to stress that we are taking the point of view of Witten in Ref. [8] that with fields of dimension +1, the perturbative expansions should be performed about a zero field background. We will not consider expansions about nontrivial backgrounds.

The paper is organized as follows. After this introduction we devote Sec. II to the construction of the most general local quadratic diff-invariant action in four dimensions (under some mild restrictions). The field equations derived from this action are linear and can be completely solved, this is the purpose of Sec. III where we do this in a systematic way. The solutions depend on a series of arbitrary elements of two different types: some of them correspond to the gauge symmetries present in the action whereas some others label non-gauge-equivalent solutions. In order to disentangle their role one must use the symplectic structure, this is done in Sec. IV. It is very important to realize that without the use of the symplectic two form it is not possible to tell which of the arbitrary parameters are gauge and which are not. Once the symplectic structure is obtained the identification of the physical degrees of freedom and gauge symmetries is straightforward. We discuss this and give some examples in Sec. V. We end the paper with several comments and our conclusions (Sec. VI). Some details of the computations are left to the Appendixes.

II. THE ACTION

As we have already stated in the Introduction, the main goal of this paper is to search for local degrees of freedom in quadratic diff-invariant actions, so our first task will be to write the most general action of this type under some restrictions. In particular we will demand the following.

(i) Absence of background structures: All the fields appearing in the action must be treated as dynamical.

(ii) Locality: The action must be local in the fields used to define it. This is arguably the most stringent condition that we impose.

(iii) The action must be, at most, quadratic in all the fields.

The previous assumptions strongly constrain the possible form of the action. By combining the absence of background structures with its quadratic character we arrive at the conclusion that the only derivative operator that can possible appear is the exterior derivative acting on differential forms. Covariant derivatives of other types of tensor fields cannot be used as they would involve quadratic terms that would force them to appear as total divergences in the action.[4] For the same reason derivatives can only act on differential forms (no other types of tensors can appear). Purely quadratic terms involving no derivatives can also be introduced; in particular, pairs of tensor fields with matching covariant and contravariant indices and total density weight +1. Let us consider $\mathcal{M}$, a four-dimensional, orientable differentiable manifold with the topology of $\mathbb{R}\times\Sigma$ where Σ is a three-dimensional compact, orientable manifold without boundary. We make a distinction between two types of fields; type 1 fields are those on which the derivative operator acts (either directly or after integration by parts); type 2 fields are not acted upon by the exterior differential operator but couple to type 1 fields. The dynamical (real) fields that will appear in the first part of the action that we introduce below are

$$\overset{1}{\varphi},\ \overset{2}{\varphi},\ \overset{1}{A},\ \overset{2}{A},\ \overset{1}{B},\ \overset{2}{B},\ \overset{1}{C},\ \overset{2}{C},\quad \text{and}\ \ D.$$

The φ fields are 0 forms, A fields are 1 forms, B fields are 2 forms, C fields are 3 forms, and D is a 4 form; they may carry internal indices, so we use the convention of taking them as column vectors and use the transpose (that we denote with a dagger even though we are dealing with real fields) whenever necessary. Ω_{11}, Ω_{12}, Ω_{22}, Θ_{11}, Θ_{12}, Θ_{21}, Θ_{22}, Σ_1, Σ_2, and Γ are constant matrices (coupling constants) with number of rows and columns–that we do not need to specify–determined by the range of the internal indices of the fields that they couple. Let us consider, then the following action:

$$S=\int_{\mathcal{M}}\Bigg[d\overset{1}{A}{}^{\dagger}\wedge\overset{1}{B}+\frac{1}{2}\overset{1}{B}{}^{\dagger}\wedge\Omega_{11}\overset{1}{B}+\overset{1}{B}{}^{\dagger}\wedge\Omega_{12}\overset{2}{B}+\frac{1}{2}\overset{2}{B}{}^{\dagger}\wedge\Omega_{22}\overset{2}{B}$$
$$+d\overset{1}{\varphi}{}^{\dagger}\wedge\overset{1}{C}+\overset{1}{A}{}^{\dagger}\Theta_{11}\overset{1}{C}+\overset{1}{A}{}^{\dagger}\wedge\Theta_{12}\overset{2}{C}+\overset{2}{A}{}^{\dagger}\wedge\Theta_{21}\overset{1}{C}$$
$$+\overset{2}{A}{}^{\dagger}\wedge\Theta_{22}\overset{2}{C}+\overset{1}{\varphi}{}^{\dagger}\Sigma_1 D+\overset{2}{\varphi}{}^{\dagger}\Sigma_2 D+\Gamma^{\dagger}D\Bigg]+\int_{\mathcal{M}}\hat{T}\cdots\tilde{S}\cdots. \tag{2}$$

Notice that we do not need to include additional coupling matrices in the derivative terms because they can be written as in Eq. (2) by a linear redefinition and the convention that fields that do no couple to derivative terms are classified as type 2. Specifically, if $\Psi\in\mathcal{M}_{M\times N}(\mathbb{R})$ and we have $dA^{\dagger}\wedge\Psi B$, we can introduce bases for $\mathbb{R}^N$ and $\mathbb{R}^M$ as $\mathcal{B}_B=\{v_1,\ldots,v_r,\rho_1,\ldots,\rho_{N-r}\}$, $\mathcal{B}_A=\{w_1,\ldots,w_r,\lambda_1,\ldots,\lambda_{M-r}\}$, where $r=\mathrm{rank}(\Psi)$, $\Psi\rho_k=0$ for $k=1,\ldots,N-r$, $\lambda_j^{\dagger}\Psi=0$ for $j=1,\ldots,M-r$ and the v's and w's are chosen so that $\mathcal{B}_A$ and $\mathcal{B}_B$ are actually bases for the corresponding vector spaces. We have now

$$dA^{\dagger}\wedge\Psi B=dA^{\dagger}(\mathcal{B}_A^{-1})^{\dagger}\mathcal{B}_A^{\dagger}\Psi\mathcal{B}_B\mathcal{B}_B^{-1}B, \tag{3}$$

where $\mathcal{B}_A^{\dagger}\Psi\mathcal{B}_B$ has the following block form

$$\begin{bmatrix}[w_a^{\dagger}\Psi v_b] & 0\\ 0 & 0\end{bmatrix}, \tag{4}$$

$[w_a^{\dagger}\Psi v_b]\in\mathcal{M}_{r\times r}(\mathbb{R})$ and is regular so that by independent linear redefinitions in A and B it can be taken to be the identity. By using the convention that fields that do not couple to derivatives are ''type 2'' we see that the derivative terms can be taken as in Eq. (2) with all generality and in

[4]We discard them by working with manifolds without boundary; we discuss the introduction of boundaries in Appendix A.

particular, that the number of internal components in $\overset{1}{A}$ and $\overset{1}{B}$ can be taken to be the same.[5]

The last term in the action (2) is built out of fields that do not couple either directly nor indirectly with the type 1, 2 fields. These fields can be any pair of tensor densities with matching (covariant and contravariant) tangent space indices and any kind of internal indices (with the corresponding coupling matrices). As we will show in due time these terms can be treated separately and it is straightforward to see that they do not describe neither local nor topological degrees of freedom.

Our purpose now is to study the dynamical content of the action introduced above, specifically we want to describe its physical degrees of freedom and gauge symmetries. A possible line of attack to this problem would be to use Hamiltonian methods (Dirac analysis of constraints and so on). This can be done in principle but is very messy in practice. The reason is that the process of finding secondary constraints is complicated by the fact that they can (and do) appear as consistency conditions in the equations that determine the Lagrange multipliers introduced in Dirac's method. As we impose no regularity conditions on the coupling matrices the process gets quite involved.

Fortunately there are other methods available that are specifically suited for the case we are considering here, these are the covariant methods considered by Witten and Crnkovic [10]. In essence they consist on working directly on the solution space to the field equations and build, from the action, the symplectic structure on this space. If one has a complete characterization of the solutions depending on arbitrary parameters (that may be fields) one can compute the restriction of the symplectic form to the solution subspace, the gauge directions (degenerate directions of the symplectic form on the solutions) and identify the physical degrees of freedom. This is usually difficult to achieve for interacting Lagrangians but can be easily done for the action (2).

III. SOLVING THE FIELD EQUATIONS

The field equations derived from Eq. (2) can be written in compact form as

$$\Sigma^{\dagger}\varphi + \Gamma^{\dagger} = 0, \tag{5}$$

$$\overset{1}{P}_{\varphi} d\varphi + \Theta^{\dagger} A = 0, \tag{6}$$

$$\overset{1}{P}_{A} dA + \Omega B = 0, \tag{7}$$

$$\overset{1}{P}_{B} dB + \Theta C = 0, \tag{8}$$

$$\overset{1}{P}_{C} dC - \Sigma D = 0 \tag{9}$$

plus the equations derived from the $\hat{T}\cdots\tilde{S}\cdots$ terms. We have used the following compact notation:

$$\Omega \equiv \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^{\dagger} & \Omega_{22} \end{bmatrix}, \quad \Theta \equiv \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \quad \Sigma \equiv \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix},$$

$$\varphi \equiv \begin{bmatrix} \overset{1}{\varphi} \\ \overset{2}{\varphi} \end{bmatrix}, \quad A \equiv \begin{bmatrix} \overset{1}{A} \\ \overset{2}{A} \end{bmatrix}, \quad B \equiv \begin{bmatrix} \overset{1}{B} \\ \overset{2}{B} \end{bmatrix}, \quad C \equiv \begin{bmatrix} \overset{1}{C} \\ \overset{2}{C} \end{bmatrix},$$

and $\overset{1}{P}_{\varphi}$, $\overset{1}{P}_{A}$, $\overset{1}{P}_{B}$, $\overset{1}{P}_{C}$ are projectors on the 1 part of φ, A, B, C

$$\begin{bmatrix} \overset{1}{\varphi} \\ \overset{2}{\varphi} \end{bmatrix} \overset{\overset{1}{P}_{\varphi}}{\mapsto} \begin{bmatrix} \overset{1}{\varphi} \\ 0 \end{bmatrix}, \begin{bmatrix} \overset{1}{A} \\ \overset{2}{A} \end{bmatrix} \overset{\overset{1}{P}_{A}}{\mapsto} \begin{bmatrix} \overset{1}{A} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \overset{1}{B} \\ \overset{2}{B} \end{bmatrix} \overset{\overset{1}{P}_{B}}{\mapsto} \begin{bmatrix} \overset{1}{B} \\ 0 \end{bmatrix}, \begin{bmatrix} \overset{1}{C} \\ \overset{2}{C} \end{bmatrix} \overset{\overset{1}{P}_{C}}{\mapsto} \begin{bmatrix} \overset{1}{C} \\ 0 \end{bmatrix},$$

respectively. The previous equations can be solved in successive steps starting from Eq. (5).

The solution to Eq. (5) is

$$\varphi(x) = \lambda_{\sigma}^{\Sigma}\varphi^{\sigma}(x) - \Sigma_{-1}^{\dagger}\Gamma^{\dagger}, \tag{10}$$

where $(\lambda_{\sigma}^{\Sigma})^{\dagger}\Sigma = 0$ and σ labels a linearly independent set of left zero eigenvectors of Σ, $-\Sigma_{-1}^{\dagger}\Gamma^{\dagger}$ denotes a particular solution to the equation $\Sigma^{\dagger}\varphi = -\Gamma^{\dagger}$, $\varphi^{\sigma}(x)$ are—at this stage—arbitrary functions, and finally, Γ must be subject to the consistency condition $\Gamma\rho_{s}^{\Sigma} = 0$ (where $\Sigma\rho_{s}^{\Sigma} = 0$ and s labels a linearly independent set of right zero eigenvectors of Σ).

Plugging Eq. (10) into Eq. (6) we get

$$\overset{1}{P}_{\varphi}\lambda_{s}^{\Sigma} d\varphi^{s}(x) + \Theta^{\dagger}A(x) = 0. \tag{11}$$

Equation (11) gives the consistency condition

$$[(\rho_{c}^{\Theta})^{\dagger}\overset{1}{P}_{\varphi}\lambda_{\sigma}^{\Sigma}] d\varphi^{\sigma}(x) \equiv \mathcal{M}_{c\sigma} d\varphi^{\sigma}(x) = 0 \tag{12}$$

(where $\Theta\rho_{c}^{\Theta} = 0$ and c labels right zero eigenvectors) and allows us to solve for $A(x)$ from Eq. (11) when Eq. (12) holds

$$A(x) = -[\Theta_{-1}^{\dagger}\overset{1}{P}_{\varphi}\lambda_{\sigma}^{\Sigma}] d\varphi^{\sigma}(x) + \lambda_{\theta}^{\Theta}A^{\theta}(x). \tag{13}$$

In analogy with the previous steps the first term of the right hand side of Eq. (13) is just a particular solution to Eq. (11) and $\lambda_{\theta}^{\Theta}$ satisfy $(\lambda_{\theta}^{\Theta})^{\dagger}\Theta = 0$. $A^{\theta}(x)$ are, at the moment, arbitrary one forms. Equation (13) will be used in the next step of process of solving the equations. To complete this step we

[5]A similar argument applies to C and φ.

need to solve Eq. (12). The dimensions of the matrix $\mathcal{M}_{c\sigma}$ are determined by the dimensions of $\ker_R\Theta$ and $\ker_L\Sigma$. To solve Eq. (12) we expand

$$\varphi^{\sigma}(x)=\varphi^{p_0}(x)[\rho^{\mathcal{M}}_{p_0}]^{\sigma}+\varphi^{q_0}(x)[v^{\mathcal{M}}_{q_0}]^{\sigma}, \tag{14}$$

where $[\rho^{\mathcal{M}}_{p_0}]^{\sigma}$, $p_0=1,\dots,\dim\ker_R \mathcal{M}$ are a complete set of right zero eigenvectors labeled by p_0; here σ explicitly labels the rows of each of these vectors. The $[v^{\mathcal{M}}_{q_0}]^{\sigma}$ are vectors in the orthogonal complement of $\ker_R \mathcal{M}$ labeled by q_0. Together with $\rho^{\mathcal{M}}_{p_0}$ they form a basis of $\mathbb{R}^{\dim\ker_L \Sigma}$.

Introducing Eq. (14) in Eq. (12) we are left with the equation

$$\mathcal{M}_{c\sigma}[v^{\mathcal{M}}_{q_0}]^{\sigma}d\varphi^{q_0}(x)=0. \tag{15}$$

As the restriction of $\mathcal{M}_{c\sigma}$ to the vector subspace generated by $\{v^{\mathcal{M}}_{q_0}\}$ is nonsingular the previous equation implies $d\varphi^{q_0}(x)=0$ and hence

$$\varphi^{q_0}(x)=f^{q_0 i_0}\varphi_{i_0}(x), \tag{16}$$

where $f^{q_0 i_0}$ are arbitrary real numbers; $\{\varphi_{i_0}\}_{i_0=1}^{\dim H^0(\mathcal{M})}\subset H^0(\mathcal{M})$ (zero de Rham cohomology group of $\mathcal{M}$; see Appendix B for a brief introduction to the de Rham cohomology) form a basis; that is i_0 goes from 1 to the number of connected components of $\mathcal{M}$. Wrapping up the previous results we conclude that

$$\varphi(x)=\{f^{q_0 i_0}\varphi_{i_0}(x)+\varphi^{p_0}(x)[\rho^{\mathcal{M}}_{p_0}]^{\sigma}\}\lambda^{\Sigma}_{\sigma}. \tag{17}$$

We also have a partial solution to $A(x)$ given by Eq. (13) that we need in order to continue with the resolution process. We leave the details for Appendix C; here we just give the final result

$$\varphi(x)=\{f^{q_0 i_0}\varphi_{i_0}(x)+\varphi^{p_0}(x)[\rho^{\mathcal{M}}_{p_0}]^{\sigma}\}\lambda^{\Sigma}_{\sigma}-\Sigma^{\dagger}_{-1}\Gamma^{\dagger}, \tag{18}$$

$$A(x)=-\{\Theta^{\dagger}_{-1}\overset{1}{P}_{\varphi}\lambda^{\Sigma}_{\sigma}\}[\rho^{\mathcal{M}}_{p_0}]^{\sigma}d\varphi^{p_0}(x)+\{A^{p_1}(x)[\rho^{\mathcal{N}}_{p_1}]^{\theta}+(\alpha^{q_1 i_1}A_{i_1}(x)+d\varpi^{q_1}_0(x))[v^{\mathcal{N}}_{q_1}]^{\theta}\}\lambda^{\Theta}_{\theta}, \tag{19}$$

$$B(x)=-\{\Omega^{\dagger}_{-1}\overset{1}{P}_{A}\lambda^{\Theta}_{\theta}\}[\rho^{\mathcal{N}}_{p_1}]^{\theta}dA^{p_1}(x)+\{B^{p_2}(x)[\lambda^{\mathcal{N}}_{p_2}]^{w}+(\beta^{q_2 i_2}B_{i_2}(x)+d\varpi^{q_2}_1(x))[v^{\mathcal{N}\dagger}_{q_2}]^{w}\}\rho^{\Omega}_{w}, \tag{20}$$

$$C(x)=-\{\Theta_{-1}\overset{1}{P}_{B}\rho^{\Omega}_{w}\}[\lambda^{\mathcal{N}}_{p_2}]^{w}dB^{p_2}(x)+\{C^{p_3}(x)[\lambda^{\mathcal{M}}_{p_3}]^{c}+(\gamma^{q_3 i_3}C_{i_3}(x)+d\varpi^{q_3}_2(x))[v^{\mathcal{M}}_{q_3}]^{c}\}\rho^{\Theta}_{c}, \tag{21}$$

$$D(x)=\{\Sigma_{-1}\overset{1}{P}_{C}\rho^{\Theta}_{\theta}\}[\lambda^{\mathcal{M}}_{p_3}]^{c}dC^{p_3}(x)+D^{s}(x)\rho^{\Sigma}_{s}. \tag{22}$$

In the previous examples the only matrices needed are $\mathcal{M}_{c\sigma}$, $\mathcal{N}_{w\theta}$ and their transposes as a consequence of the fact that

$$\mathcal{M}_{c\sigma}=(\rho^{\Theta}_{c})^{\dagger}\overset{1}{P}_{\varphi}\lambda^{\Sigma}_{\sigma}=(\lambda^{\Lambda}_{\sigma})^{\dagger}\overset{1}{P}_{C}\rho^{\Theta}_{c}=\mathcal{M}^{\dagger}_{\sigma c},$$
$$\mathcal{N}_{w\theta}=(\rho^{\Omega}_{w})^{\dagger}\overset{1}{P}_{A}\lambda^{\Theta}_{\theta}=(\lambda^{\Theta}_{\theta})^{\dagger}\overset{1}{P}_{B}\rho^{\Omega}_{w}=\mathcal{N}^{\dagger}_{\theta w}. \tag{23}$$

We have used the notation explained in Appendix C. Here we just discuss the general features of Eqs. (18)–(22). First we notice that the solution is parametrized by three types of objects: constant parameters $f^{q_0 i_0}$, $\alpha^{q_1 i_1}$, $\beta^{q_2 i_2}$, $\gamma^{q_3 i_3}$ that multiply elements of the bases of cohomology groups $H^0(\mathcal{M}),\dots,H^3(\mathcal{M})$ and two sets of differential forms $\varphi^{p_0}(x)$, $A^{p_1}(x)$, $B^{p_2}(x)$, $C^{p_3}(x)$, $D^{s}(x)$ and $\varpi^{q_1}_0(x)$, $\varpi^{q_2}_1(x)$, $\varpi^{q_3}_2(x)$ [or rather $d\varpi^{q_1}_0(x)$, $d\varpi^{q_2}_1(x)$, $d\varpi^{q_3}_2(x)$]. It is important to emphasize that we are not entitled to disregard neither $A^{p_1}(x)$, $B^{p_2}(x)$, $C^{p_3}(x)$, $D^{s}(x)$ nor $\varpi^{q_1}_0(x)$, $\varpi^{q_2}_1(x)$, and $\varpi^{q_3}_0(x)$ without knowing that they are gauge parameters; a fact that we *ignore* at this stage.

IV. THE SYMPLECTIC STRUCTURE

Given a quadratic action the problem of finding out its gauge symmetries is intimately connected with that of solving the Euler-Lagrange equations. If we formally write a quadratic action for a set of fields $\{\phi^i\}_{i\in I}$

$$S[\phi]=\int \phi^i A_{ij}\phi^j, \tag{24}$$

and A a field independent symmetric linear operator, the variation of S when we change ϕ^i by $\phi^i+\delta\phi^i$ is given by the exact expression

$$\delta S=\int[\delta\phi^i A_{ij}\phi^j+\phi^i A_{ij}\delta\phi^j+\delta\phi^i A_{ij}\delta\phi^j], \tag{25}$$

whereas the Euler-Lagrange equations are $A_{ij}\phi^j=0$. If χ^i is such that $A_{ij}\chi^j=0$ and we consider *any* field configuration ϕ^i the action is invariant under the transformation $\phi^i\mapsto\phi^i+\chi^i$ as can be seen from Eq. (25). Of course, in nongauge invariant theories, this is just the linear superposition principle. A gauge symmetry would manifest itself in a similar way but now it should be thought of as an arbitrariness in a set of solutions corresponding, loosely speaking, to the same initial conditions.[6] The question is then: How do we discriminate between both situations?

One could be tempted to think that if we happen to have a complete parametrization of the solutions to the field equations, depending on a certain set of parameters (that can actually be functions and should, at least, describe all the possible initial conditions) it is straightforward to tell gauge parameters apart from the other quantities needed to specify

[6]One should take into account the possibility of using different initial data hypersurfaces.

a solution. It is very important to understand that it is not the case. If we are given a subset of a space field configurations parametrized by certain functions there is no way to decide, in a uniquely consistent manner, which of them are gauge parameters and which ones parametrize inequivalent configurations. In order to do that an extra element is needed: a suitably defined symplectic form in the space of fields. Though it is more frequent to find the symplectic structure in the Hamiltonian formulation there are covariant methods [10] that allow us to work directly with field configurations. The necessity of a symplectic form is clear in the Hamiltonian framework. Given a phase space and certain submanifold of it (let us suppose that the Hamiltonian is zero) we need the symplectic form ω to define what we understand as gauge orbits and gauge transformations (this is done by looking at degenerate directions of ω on the constraint hypersurface). The situation is analogous in the covariant setting.

In sight of Eqs. (18)–(22) one is tempted to conclude that the action (2) just describes the topological degrees of freedom labeled by $f^{q_0 i_0}, \alpha^{q_1 i_1}, \beta^{q_2 i_2}, \gamma^{q_3 i_3}$. In simple examples it is possible, in practice, to directly write down a set of independent gauge transformations and, actually guess, the physical degrees of freedom. This is not that easy here and a different method must be used. In our case, and following [10], the symplectic structure is given by

$$\omega = \int_\Sigma J = \int_\Sigma [\mathbb{d}\overset{1}{A}{}^\dagger \wedge \mathbb{d}\overset{1}{B} + \mathbb{d}\overset{1}{\varphi}{}^\dagger \wedge \mathbb{d}\overset{1}{C}]. \tag{26}$$

In the previous expressions $\mathbb{d}$ and $\wedge$ denote the exterior differential and product in the field space *coordinatized* by $\varphi(x)$, $A(x)$, $B(x)$, $C(x)$, and $D(x)$. The exterior product in $\mathcal{M}$ is implicitly understood in Eq. (26). In order to perform the three dimensional integral appearing in Eq. (26) one must specify the three manifold Σ. That the result is independent of Σ is a consequence of the fact that $dJ=0$ on solutions to the field equations (here d is the exterior differential on $\mathcal{M}$). This can be checked in a straightforward way by acting on J with d; keeping track of the form order both in $\mathcal{M}$ and in the field space (to get the minus signs right) and using the field equations (5)–(9) after explicitly splitting them in type 1 and type 2 fields.

Once we have ω we must compute it on the solutions to the fields equations given by Eqs. (18)–(22), as a previous step to finding the degenerate directions. This way we get

$$\omega|_{\text{sol}} = [v^{\mathcal{M}}_{q_0}]^\sigma \mathcal{M}^\dagger_{\sigma c} [v^{\mathcal{M}^\dagger}_{q_3}]^c \left(\int_\Sigma \phi_{i_0} C_{i_3} \right) \mathbb{d} f^{q_0 i_0} \wedge \mathbb{d}\gamma^{q_3 i_3} + [v^{\mathcal{N}}_{q_1}]^\theta \mathcal{N}^\dagger_{\theta w} [v^{\mathcal{N}^\dagger}_{q_2}]^w \left(\int_\Sigma A_{i_1} \wedge B_{i_2} \right) \mathbb{d}\alpha^{q_1 i_1} \wedge \mathbb{d}\beta^{q_2 i_2}. \tag{27}$$

Several points are now in order. First of all the fact that the only parameters that appear in $\omega|_{\text{sol}}$ are $f^{q_0 i_0}$, $\alpha^{q_1 i_1}$, $\beta^{q_2 i_2}$, and $\gamma^{q_3 i_3}$ show that the theory has no local degrees of freedom ($\omega|_{\text{sol}}$ acting on tangent vectors to the solution manifold is zero whenever these vectors ''point in the directions'' parametrized by all the x-dependent fields). Second, even though one would naively expect that the only parameters appearing in Eq. (27) are those corresponding to the ''topological sectors'' the fact that x-dependent fields do not appear in Eq. (27) depends on nontrivial cancellations between terms coming from the first wedge product in Eq. (26) and terms coming from the second (see Appendix D). Third, from the Künneth formula and the fact that the cohomology groups of $\mathbb{R}$ are $H^0(\mathbb{R}) = \mathbb{R}$ and $H^1(\mathbb{R}) = 0$ we conclude that $H^s(\mathcal{M}) = H^s(\Sigma)$, in particular notice that this is the reason why $H^4(\mathcal{M})$ does not appear. Finally, each of the two terms in Eq. (27) has a coefficient (a matrix in the pairs of indices labeling f, α, β, γ) that could, in principle, be degenerate. A fact that would imply the presence of additional gauge transformations involving now the topological sectors. This, however, is not the case. On one hand both integrals

$$\int_\Sigma \varphi_{i_0} C_{i_3}; \quad \int_\Sigma A_{i_1} \wedge B_{i_2} \tag{28}$$

are nonsingular square matrices (see Appendix B). On the other both

$$[v^{\mathcal{M}}_{q_0}]^\sigma \mathcal{M}^\dagger_{\sigma c} [v^{\mathcal{M}^\dagger}_{q_3}]^c, \quad [v^{\mathcal{N}}_{q_1}]^\theta \mathcal{N}^\dagger_{\theta w} [v^{\mathcal{N}^\dagger}_{q_2}]^w \tag{29}$$

are easily proved to be square nonsingular matrices so that we finally conclude that Eq. (27) contains no degenerate directions. Taking appropriate bases in the cohomology groups it is straightforward to write Eq. (27) in canonical form.

At this point it is a simple matter to deal with the additional $\hat{T}\cdots\tilde{S}\cdots$ terms in the original action. As they do not involve derivatives they do not appear in the symplectic structure (and its restriction to the solution subspace) which means that all the degrees of freedom described by them are pure gauge.

V. DEGREES OF FREEDOM AND GAUGE TRANSFORMATIONS

We summarize our previous results from the last sections. As we have just seen the only physical degrees of freedom described by Eq. (2) are purely topological and described by pairs of variables $(f^{q_0 i_0}, \gamma^{q_3 i_3})$ and $(\alpha^{q_1 i_1}, \beta^{q_2 i_2})$. The gauge symmetries of the action (2) are

$$\delta\varphi(x) = \delta\varphi^{p_0}(x) [\rho^{\mathcal{M}}_{p_0}]^\sigma \lambda^\Sigma_\sigma, \tag{30}$$

$$\delta A(x) = -\{\Theta^\dagger_{-1} \overset{1}{P}_\varphi \lambda^\Sigma_\sigma\} [\rho^{\mathcal{M}}_{p_0}]^\sigma d\,\delta\varphi^{p_0}(x) + \{\delta A^{p_1}(x) [\rho^{\mathcal{N}}_{p_1}]^\theta + d\,\delta\varpi^{q_1}_0(x) [v^{\mathcal{N}}_{q_1}]^\theta\} \lambda^\Theta_\theta,$$

$$\delta B(x) = -\{\Omega^\dagger_{-1} \overset{1}{P}_A \lambda^\Theta_\theta\} [\rho^{\mathcal{N}}_{p_1}]^\theta d\,\delta A^{p_1}(x) + \{\delta B^{p_2}(x) [\lambda^{\mathcal{N}}_{p_2}]^w + d\,\delta\varpi^{q_2}_1(x) [v^{\mathcal{N}^\dagger}_{q_2}]^w\} \rho^\Omega_w,$$

$$\delta C(x) = -\{\Theta_{-1}\overset{1}{P}_B\rho_w^{\Omega}\}[\lambda_{p_2}^{\mathcal{N}}]^w d\,\delta B^{p_2}(x) + \{\delta C^{p_3}(x)[\lambda_{p_3}^{\mathcal{M}}]^c + d\,\delta\varpi_2^{q_3}(x)[v_{q_3}^{\mathcal{M}}]^c\}\rho_c^{\Theta},$$

$$\delta D(x) = \{\Sigma_{-1}\overset{1}{P}_C\rho_\theta^{\Theta}\}[\lambda_{p_3}^{\mathcal{M}}]^c d\,\delta C^{p_3}(x) + \delta D^s(x)\rho_s^{\Sigma},$$

where the gauge parameters are the 0 forms $\delta\varphi^{p_0}(x), \delta\varpi_0^{q_1}(x)$, the one forms $\delta A^{p_1}(x), \delta\varpi_1^{q_2}(x)$, the two forms $\delta B^{p_2}(x), \delta\varpi_2^{q_3}(x)$, the three forms $\delta C^{p_3}(x)$, and the four forms $\delta D^s(x)$. Though, *a posteriori*, the structure of these transformations is quite simple it is not straightforward to guess the complicated matrix coefficients appearing in the previous expressions.

We consider now some examples of these types of theories that have been studied in the literature [11,12]. In them some of the fields (φ and C) do not appear. This fact is taken into account in Eq. (30) by putting $P_\varphi = P_C = 0$ and realizing that $\delta\varphi$ and δC are then arbitrary and do not affect both δA and δB.

Example 1. Abelian *BF* model with no internal indices. Here

$$S = \int_{\mathcal{M}} dA \wedge B, \tag{31}$$

so that $\Omega = 0$, and the other fields do not appear. In this case $\mathcal{M}_{c\sigma} = 0$, $\mathcal{N}_{\theta w} = 1$; $v^{\mathcal{N}} = 1$, $v^{\mathcal{N}^\dagger} = 1$, then $\omega|_{\text{sol}} = (\int_\Sigma A_{i_1} \wedge B_{i_2})\,\mathbb{d}\alpha^{i_1} \wedge \mathbb{d}\beta^{i_2}$ and the theory has $\dim H^1(\Sigma) = \dim H^2(\Sigma)$ degrees of freedom.

The gauge transformations are

$$\delta A(x) = d\,\delta\varpi_0(x), \quad \delta B(x) = d\,\delta\varpi_1(x). \tag{32}$$

Example 2.

$$S = \int_{\mathcal{M}} \left[dA \wedge B - \frac{1}{2} B \wedge B \right], \tag{33}$$

so that $\Omega = -1$, $\mathcal{M}_{c\sigma} = 0$, $\mathcal{N}_{\theta w} = 0$; $v^{\mathcal{N}} = v^{\mathcal{M}} = 0$. This means that, on solutions, $\omega|_{\text{sol}} = 0$ so that all the points in the solution subspace are in the same gauge orbit and hence the theory has zero degrees of freedom. The gauge transformations are

$$\delta A(x) = \delta\chi(x), \quad \delta B(x) = d\,\delta\chi(x). \tag{34}$$

Example 3.[7]

$$S = \int_{\mathcal{M}} [dA_1 \wedge B_1 + dA_2 \wedge B_2 + B_1 \wedge B_2], \tag{35}$$

[7]This example appears as the quadratic term in the formulation of the Husain-Kuchař model as a coupled *BF* system [12].

with $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\overset{1}{P}_A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\overset{1}{P}_B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\rho^{\Omega} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $\mathcal{M}_{c\sigma} = 0$, $\mathcal{N}_{w\theta} = 0$, so that $v^{\mathcal{N}} = v^{\mathcal{M}} = 0$, $\rho_1^{\mathcal{N}} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\rho_2^{\mathcal{N}} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and, as before, the theory has zero degrees of freedom. The gauge transformations are

$$\begin{aligned} \delta A_1(x) &= \delta\chi_1(x), \quad \delta A_2(x) = \delta\chi_2(x), \\ \delta B_1(x) &= -d\,\delta\chi_2(x), \quad \delta B_2(x) = -d\,\delta\chi_1(x). \end{aligned} \tag{36}$$

Notice how the switch in the roles of $\delta\chi_1(x)$ and $\delta\chi_2(x)$ in the gauge transformations for the $B_1(x), B_2(x)$ fields is correctly described by the general form of the gauge transformations given by $\delta B(x)$ in Eq. (28).

VI. CONCLUSIONS AND COMMENTS

At this point, and after realizing that the final phase space is that of a set of uncoupled *BF* and $C\varphi$ theories, the attentive reader may wonder if it would not be easier to simplify the action from the start by solving for the purely algebraic field equations and substituting the solution back in the action in order to eliminate, for example, the type 2 fields and work with a simple set of *BF* and $C\varphi$ theories. Though, in actual examples, this may be a useful strategy there is a catch. The algebraic field equations usually allow to solve only for combinations of type 1 and type 2 fields so that, in the end, one may not find a much simpler action. That this is so is actually proved by the rather tangled nature of the gauge transformations (30) and specially by the awkward matrices that appear in the solution. A different manifestation of the same problem appears if one tries to work within the Hamiltonian framework.

The type of theory described by the action considered above may be taken as the building block of more complicated topological and nontopological actions obtained from Eq. (2) by adding higher power terms to it. Their symmetries must be understood, among other things, in order to treat them perturbatively (see, for example, Ref. [13], and references therein). This is an additional reason to work with an action of the general form introduced above, because it is not clear that a nonlinear extension of it can be equally obtained from a simplified action.

It is straightforward to extend the previous analysis to arbitrary dimensions and arrive at the conclusion that local diff-invariant quadratic Lagrangians in arbitrary space-time dimensions do not have local degrees of freedom. This means that in order to describe local degrees of freedom with local diff-invariant Lagrangians they must be taken to be, at least, cubic in the fields. This happens, for example, if the background metric used to write down ordinary free actions is taken as a dynamical object.

It cannot be overemphasized that without the use of the symplectic structure there is no way of telling which one of the parameters and functions appearing in the general solution to the field equations are gauge and which ones label genuine degrees of freedom. We feel that the conclusion that no local degrees of freedom survive is not obvious *a priori*. An important consequence of this result is that there are no diff-invariant ''lower derivative'' modifications of the action (1) that can be treated perturbatively. Notice, however, that our result says nothing against the existence of a similar possibility in Palatini-like actions where quadratic and cubic terms are not separately diff-invariant.

Another interesting consequence of the previous analysis is related to the meaning of the particle concept in diff-invariant theories. In ordinary quantum field theories in a Minkowskian background the study of particle states is done by first choosing a quadratic (''free'') Lagrangian, finding its normal mode expansion and labeling the resulting ''elementary excitations'' with momentum and spin indices [coming from the Casimir operators of the space-time symmetry group ISO(1,3)]. If one has a quadratic theory with local degrees of freedom in a non Minkowskian background (as one does in QFT's in curved backgrounds) one still has the normal mode decomposition but lacks the Poincaré symmetry. In this case, though the particle interpretation is subtler (coordinate dependence, Unruh effect and so on) there is still one available. What we have found here is that in the absence of a background there is no possible particle interpretation. This result lends support to some approaches to dealing with diff-invariant QFT's such as the loop variables approach to quantum gravity championed by Ashtekar, Rovelli, Smolin, and others [5,6], and in particular helps understand why the elementary excitations (spin network-states and so on) are so different from the usual Fock-space particle states. It may also be worthwhile to point out that string theory and its multiple derivatives require the presence of a nondynamical background.

Finally one may wonder how it would be possible to escape the negative conclusions of this paper. If one is willing to keep diffeomorphism invariance and the quadratic character of the Lagrangian one would be forced to abandon locality. If a nonlocal, diff-invariant quadratic action describing propagating degrees of freedom exists is an open problem that may be worth thinking about.

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APPENDIX A: INCLUSION OF BOUNDARIES

The result presented in the paper that quadratic, local diff-invariant theories, have no local degrees of freedom has been derived in the case of working with manifolds without a boundary. The attentive reader may wonder if the result still holds in the presence of boundaries. The answer is in the affirmative as justified in what follows. If $\mathcal{M}$ has boundaries the action (2) can be generalized by adding the most general diff-invariant action on its boundary $\partial\mathcal{M}$

$$S_\partial=\int_{\partial\mathcal{M}}\Big[d\overset{1}{a}{}^\dagger\wedge\overset{1}{a}+d\overset{1}{\phi}{}^\dagger\wedge\overset{1}{b}+\overset{1}{a}{}^\dagger\wedge\theta_{11}\overset{1}{b}+\overset{1}{a}{}^\dagger\wedge\theta_{12}\overset{2}{b}+\overset{2}{a}{}^\dagger\wedge\theta_{21}\overset{1}{b}+\overset{2}{a}{}^\dagger\wedge\theta_{22}\overset{2}{b}+\overset{1}{\phi}{}^\dagger\sigma_1 c+\overset{2}{\phi}{}^\dagger\sigma_2 c+\gamma^\dagger c\Big], \tag{A1}$$

where $\overset{1}{\phi}, \overset{2}{\phi}$ are 0 forms, $\overset{1}{a}, \overset{2}{a}$ are 1 forms, $\overset{1}{b}, \overset{2}{b}$ are 2 forms, and c is a 3 form. One can think of the terms in S_∂ as having two different origins; some of them come from the restriction of the four-dimensional fields to $\partial\mathcal{M}$ whereas others are genuine three-dimensional fields. If one adopts this point of view the field equations are obtained by varying first in the four-dimensional fields with variations restricted to be zero on the boundary and then varying the three-dimensional fields. Another possible, and equivalent, point of view would be to consider that a part of S_∂ provides the surface terms needed to cancel the surface terms appearing when we vary the four-dimensional fields and integrate by parts whereas the rest of S_∂ consists of the genuine three-dimensional objects. In either case we are entitled to treat four-dimensional and three-dimensional fields independently, so that in order to prove that no local degrees of freedom appear when boundaries are present it suffices to show that the action (A1) (defined on the boundary of $\mathcal{M}, \partial\mathcal{M}=\mathbb{R}\times\partial\Sigma$, with $\partial^2\Sigma=\varnothing$ as a consequence of the identity $\partial^2=0$ for the compact two-manifold $\partial\Sigma$) describes no local degrees of freedom. This is done by following exactly the same steps presented in the main body of the paper for the four-dimensional case so we do not give the details here.

APPENDIX B: de RHAM COHOMOLOGY GROUPS

For the benefit of the physics-oriented reader we summarize here the main results about differential forms and the de Rham cohomological groups. A manifold $\mathcal{M}$ without (with) boundary of dimension m is a locally m-Euclidean (semi-m-Euclidean) topological space. Smooth manifolds are those that are locally Euclidean in a smooth way.

All smooth manifolds (with or without boundary) admit smooth tensor fields. In particular, the completely antisymmetric r-covariant tensor fields are known as r-differential forms. Forms can be differentiated by means of the exterior differential operator d, that maps r forms in $(r+1)$ forms. The d operator has the property $d^2=0$, that allows us to define the cohomology groups on $\mathcal{M}$. A r form w on $\mathcal{M}$ is closed if $dw=0$ and is exact if $w=dv$ for some $(r-1)$-form v on $\mathcal{M}$. Exact forms are all closed because $d^2=0$. The converse is not true as one can see if one defines an equivalence relation on the vector space of closed r forms: two closed forms w_1 and w_2 are called cohomologous if w_1-w_2 is exact. The set of equivalence classes is denoted by $H^r(\mathcal{M})$, the rth de Rham cohomology group of $\mathcal{M}$. Although the cohomology groups are defined in terms of the manifold structure of $\mathcal{M}$, they are topological invariants; that is, two topologically equivalent manifolds have the same de Rham groups. For compact manifolds they are finite dimensional real vector spaces. Another property that we use in the paper is the fact that for compact manifolds H^r can be identified with $H^{(m-r)}$, this is known as Poincaré duality and is a consequence of the fact that integrals of the type given by Eq. (28) define a nonsingular bilinear form in $H^r\times H^{(m-r)}$ [15].

Directly from the most general version of the Poincaré Lema [14], that states that $H^r(\mathbb{R}\times\mathcal{M})=H^r(\mathcal{M})$, or indirectly, by means of the Künneth formula [15] that relates the cohomology groups of $\mathcal{M}=\mathcal{M}_1\times\mathcal{M}_2$ with the $\mathcal{M}_1,\mathcal{M}_2$ ones according to

$$H^r(\mathcal{M})=\oplus_{p+q=r}[H^p(\mathcal{M}_1)\otimes H^q(\mathcal{M}_2)],$$

one can prove $H^r(\mathbb{R}\times\Sigma)=H^r(\Sigma)$, an assertion that we use in Sec. IV.

APPENDIX C: SOLVING THE FIELD EQUATIONS

We continue here with the resolution of the field equations started in Sec. III. Introducing Eq. (13) in Eq. (7) we get

$$[\overset{1}{P}_A\lambda^{\Theta}_{\theta}]dA^{\theta}(x)+\Omega B(x)=0 \tag{C1}$$

which gives, as in the previous step, a consistency condition and a solution for B. The consistency condition is

$$\{(\rho^{\Theta}_w)^{\dagger}\overset{1}{P}_A\lambda^{\Theta}_{\theta}\}dA^{\theta}(x)\equiv\mathcal{N}_{w\theta}dA^{\theta}(x)=0 \tag{C2}$$

(where $\Omega\rho^{\Omega}_w=0$ and w labels these right zero eigenvectors) and we get for $B(x)$

$$B(x)=-\{\Omega_{-1}\overset{1}{P}_A\lambda^{\Theta}_{\theta}\}dA^{\theta}(x)+B^w(x)\rho^{\Omega}_w. \tag{C3}$$

The first term in the previous expression is a particular solution to Eq. (C1). Expanding

$$A^{\theta}(x)=A^{p_1}(x)[\rho^{\mathcal{N}}_{p_1}]^{\theta}+A^{q_1}(x)[v^{\mathcal{N}}_{q_1}]^{\theta} \tag{C4}$$

with $[\rho^{\mathcal{N}}_{p_1}]^{\theta}, p_1=1,\ldots,$ dim $\ker_R\mathcal{N}$ a complete generating set of right zero-eigenvectors labeled by p_1 (θ labels the rows of each of these vectors) and $[v^{\mathcal{N}}_{q_1}]^{\theta}$ a basis in the complement of ker $_R\mathcal{N}$. As before we have to solve the equation

$$dA^{q_1}(x)=0 \tag{C5}$$

which gives

$$A^{q_1}(x)=\alpha^{q_1i_1}A_{i_1}(x)+d\varpi^{q_1}_0(x). \tag{C6}$$

Notice that each $A^{q_1}(x)$ is a closed one form [an element of $Z^1(\mathcal{M})$] that we write as an element in $H^1(\mathcal{M})$ plus an arbitrary exact one-form $d\varpi^{q_1}_0(x)$ (where $\varpi^{q_1}_0$ are 0 forms). The $A_{i_1}(x)$ are a basis of elements in $H^1(\mathcal{M})$. Plugging A back in Eq. (13) gives Eq. (19).

In the next step we introduce Eq. (C3) in Eq. (8) to get

$$[\overset{1}{P}_B\rho^{\Omega}_w]dB^w(x)+\Theta C(x)=0. \tag{C7}$$

The consistence condition derived from Eq. (C7) is

$$\{(\lambda^{\Theta}_{\theta})^{\dagger}\overset{1}{P}_B\rho^{\Omega}_w\}dB^w(x)\equiv\mathcal{N}^{\dagger}_{\theta w}dB^w(x)=0 \tag{C8}$$

and

$$C(x)=-\{\Theta_{-1}\overset{1}{P}_B\rho^{\Omega}_w\}dB^w(x)+C^c(x)\rho^{\Theta}_c \tag{C9}$$

the first term in Eq. (C9) being a particular solution to Eq. (C7) and $\Theta\rho^{\Theta}_c=0$. Expanding

$$B^w(x)=B^{p_2}(x)[\lambda^{\mathcal{N}}_{p_2}]^w+B^{q_2}(x)[v^{\mathcal{N}}_{q_2}]^w \tag{C10}$$

in complete analogy with the previous steps, we obtain the equation $dB^{q_2}(x)=0$

$$B^{q_2}(x)=\beta^{q_2i_2}B_{i_2}(x)+d\varpi^{q_2}_1(x), \tag{C11}$$

where $\{B_{i_2}\}$ is a basis of $H^2(\mathcal{M})$ and $\varpi^{q_2}_1$ are one forms. Introducing Eq. (C11) back in Eq. (C3) we finally obtain Eq. (20). In the following step we introduce Eq. (C9) in Eq. (9) and obtain

$$[\overset{1}{P}_C\rho^{\Omega}_c]dC^c(x)-\Sigma D(x)=0 \tag{C12}$$

which gives the consistency condition

$$\{(\lambda^{\Sigma}_{\sigma})^{\dagger}\overset{1}{P}_C\rho^{\Omega}_c\}dC^c(x)=0 \tag{C13}$$

and

$$D(x)=\{\Sigma_{-1}\overset{1}{P}_C\rho^{\Omega}_c\}dC^c(x)+D^s(x)\rho^{\Sigma}_s. \tag{C14}$$

As in previous cases $\{\Sigma_{-1}\overset{1}{P}_C\rho^{\Omega}_c\}dC^c(x)$is a particular solution to Eq. (C12) (as an equation in D). Expanding now

$$C^c(x)=C^{p_3}(x)[\lambda^{\mathcal{M}}_{p_3}]^c+C^{q_3}(x)[v^{\mathcal{M}^{\dagger}}_{q_3}]^c \tag{C15}$$

we need to solve $dC^{q_3}(x)=0$ and hence

$$C^{q_3}(x)=\gamma^{q_3i_3}C_{i_3}(x)+d\varpi^{q_3}_2(x) \tag{C16}$$

where the $C_{i_3}(x)$ spans a basis in $H^3(\mathcal{M})$ and $\varpi^{q_3}_2$ are two forms. Equation (C16) allows us to obtain Eq. (21) from Eq. (C9) and Eq. (22) from Eq. (C14).

APPENDIX D: THE SYMPLECTIC STRUCTURE ON SOLUTIONS

In order to appreciate the nontriviality of the cancellations leading to the final symplectic structure (27) we separately compute

$$\int_{\Sigma}\mathrm{d}\overset{1}{\varphi}{}^{\dagger}\wedge\mathrm{d}\overset{1}{C}=[v^{\mathcal{M}}_{q_0}]^{\sigma}\mathcal{M}^{\dagger}_{\sigma c}[v^{\mathcal{M}^{\dagger}}_{q_3}]^c\left(\int_{\Sigma}\phi_{i_0}C_{i_3}\right)\mathrm{d}f^{q_0i_0}\wedge\mathrm{d}\gamma^{q_3i_3}$$
$$-\{(\lambda^{\Sigma}_{\sigma})^{\dagger}\overset{1}{P}_C\Theta_{-1}\overset{1}{P}_B\rho^{\Omega}_w\}[\rho^{\mathcal{M}}_{q_0}]^{\sigma}\int_{\Sigma}\mathrm{d}\varphi^{q_0}\wedge d\,\mathrm{d}B^w \tag{D1}$$

with $B^w(x)$ given by Eqs. (C10) and (C11). Notice that, in general, the last term in the right hand side of the previous expression is not necessarily zero.

$$\int_\Sigma \mathbb{d}\overset{1}{A}^\dagger \wedge \mathbb{d}\overset{1}{B} = [v^{\mathcal{N}}_{q_1}]^\theta \mathcal{N}^\dagger_{\theta w}[v^{\mathcal{N}^\dagger}_{q_2}]^w \left(\int_\Sigma A_{i_1} \wedge B_{i_2} \right) \mathbb{d}\alpha^{q_1 i_1} \wedge \mathbb{d}\beta^{q_2 i_2} + \{(\lambda^\Sigma_\sigma)^\dagger \overset{1}{P}_C \Theta_{-1} \overset{1}{P}_B \rho^\Omega_w\}[\rho^{\mathcal{M}}_{q_0}]^\sigma \int_\Sigma \mathbb{d}\varphi^{q_0} \wedge d\mathbb{d}B^w \tag{D2}$$

after manipulating the matrices appearing in the last term of Eq. (D2). In the process of obtaining Eqs. (D1) and (D2) several terms directly cancel by any of these reasons: (i) appearance of $\mathcal{M}\rho^{\mathcal{M}}_{p_0}$; (ii) $d^2=0$; (iii) Integration by parts (remember that we take $\partial\Sigma=\emptyset$); (iv) $d\varphi_{i_0}=0$.

Finally, adding up Eqs. (D1) and (D2) gives (27).

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